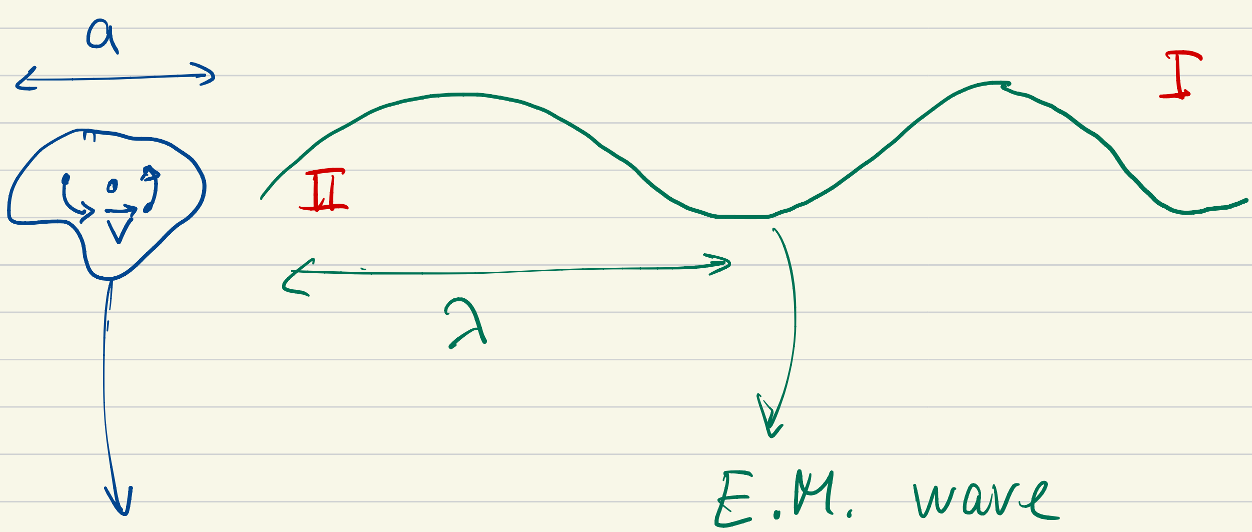
Review of Lecture 8



localized system of moving charges

E.M. wave

I: $a \ll \lambda \ll x$

II: $a \ll x \ll \lambda$

$$\omega = \frac{c}{\lambda} = \frac{v}{a}$$

I: $\frac{a}{\lambda}$ (dynamical multipole expansion) is more important than $\frac{a}{x}$ expansion

$$\vec{B} = \frac{\mu_0}{4\pi c} \frac{1}{|\vec{x}|} \ddot{\vec{d}} \times \vec{n}$$

$$\vec{E} = -c\vec{n} \times \vec{B}$$

(plane waves)

II: $\frac{a}{x}$ (static multipole expansion) is more important:

$$\varphi(\vec{x},t) =$$

2^n-pole tensor.

$$= \sum_{n=0}^{\infty} \frac{Q_{i_1 \dots i_n}(t)\, x_{i_1} \dots x_{i_n}}{|\vec{x}|^{2n+1}} \left(1 + O\left(\frac{a}{\lambda}\right)\right)$$

Lecture 9

EM fields in medium

Until now we studied EM fields in the vacuum or in the presence of smooth charge density. In reality charge and currents are localized to atoms and ions inside bodies or liquids and change dramatically on short scales. So do EM fields. Let us denote such microscopic fields in the following way:

$$\vec{E} \to \vec{e} \qquad \rho \to \eta$$

$$\vec{B} \to \vec{b} \qquad \vec{J} \to \vec{j}$$

$\vec{e}$, $\vec{b}$, ρ and $\vec{j}$ are what we used to denote by $\vec{E}$, $\vec{B}$, ρ and $\vec{J}$.

(solutions of exact Maxwell equations)

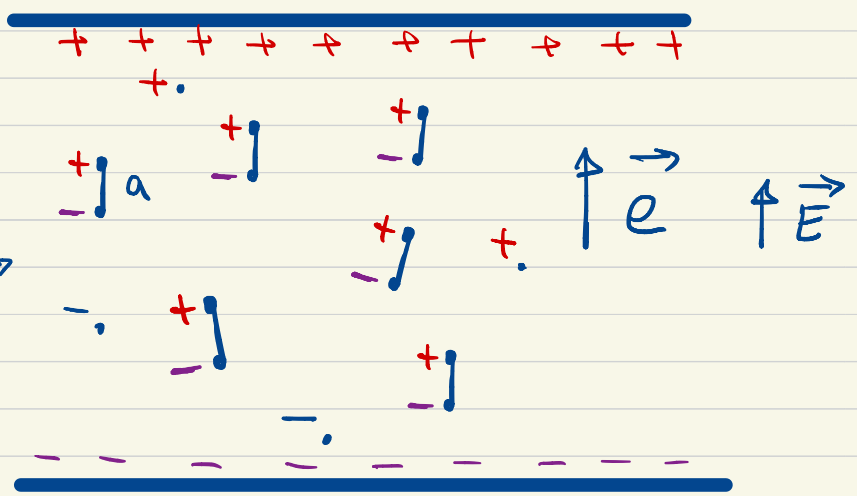
$$\vec{\nabla}\cdot\vec{e} = \frac{\eta}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{e} = -\frac{\partial \vec{b}}{\partial t}$$

$$\vec{\nabla}\times\vec{b} = \mu_0 \vec{j} + \frac{\partial \vec{e}}{\partial t}\varepsilon_0\mu_0 \qquad \vec{\nabla}\cdot\vec{b} = 0$$

Now we will change notation and denote by $\vec{E}$, $\vec{B}$, ρ and $\vec{J}$ the averaged quantities. In a medium we care about the averaged fields.

vacuum →

medium →



(partially compensated by surface charges)

vacuum →

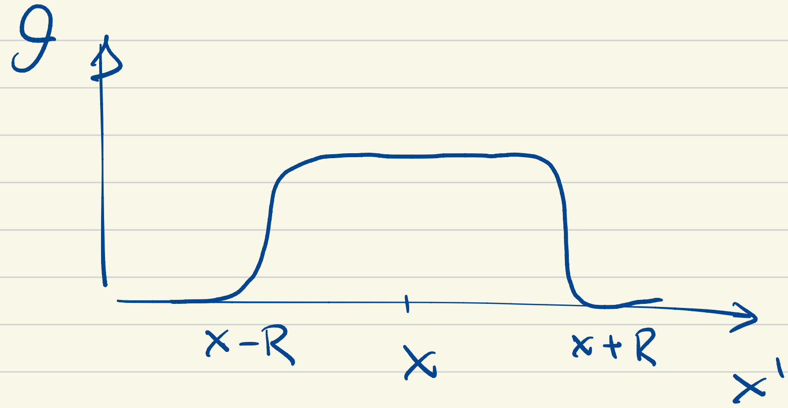
Dipols present inside the material align with $\vec{e}$: $\vec{d} \uparrow\uparrow \vec{e}$, their own field partially compensates $\vec{e}$.

[Conductors have enough free moving charges to fully screen the field]

To develope a more rigorous picture lets define an averaging "window" function $f(x'-x)$:

$$\int f(\vec{x}'-\vec{x})\, d^3x' = 1$$

$$f(\vec{x}'-\vec{x}) = 0 \ , \text{ if } \ |\vec{x}'-\vec{x}| > R$$



$R \gg a$, where $a \sim$ characteristic scale of the distribution [atomic scale]

Now we define macroscopic, or averaged fields:

$$\vec{E}(\vec{x},t) = \int d^3x' \, \vec{e}(\vec{x}',t) \, f(\vec{x}'-\vec{x}) \equiv \langle e \rangle$$

$$\vec{B}(\vec{x},t) = \int d^3x' \, \vec{b}(\vec{x}',t) \, f(\vec{x}'-\vec{x}) \equiv \langle b \rangle$$

Derivatives commute with averaging:

$$\partial_x \vec{E} = \int d^3x' \, \vec{e}(x',t) \, \partial_x f(\vec{x}'-\vec{x}) =$$

$$= -\int d^3x' \, \vec{e}(x',t) \, \partial_{x'} f(\vec{x}'-\vec{x}) =$$

$$= \int d^3x' \, \partial_{x'} \vec{e}(\vec{x}',t) \, f(\vec{x}'-\vec{x}) \implies$$

Apply averaging to Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\langle \eta \rangle}{\varepsilon_0}$$

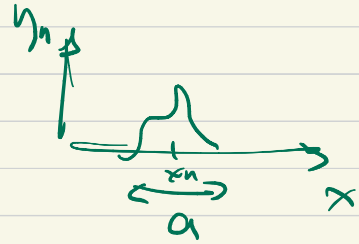
$$\vec{\nabla} \times \vec{B} = \mu_0 \langle \vec{j} \rangle + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Charge density is a sum over particles:

$$\eta(\vec{x}) = \sum_n \eta_n(x - x_n)$$



localized on atomic scales $\sim a$

$$\langle \eta \rangle = \sum_n \int d^3x' \, \eta_n(\vec{x}' - \vec{x}_n) \, f(\vec{x}' - \vec{x})$$

Taylor expand $f(\vec{x}' - \vec{x})$ at $\vec{x}' = x_n$:

$$f(\vec{x}' - \vec{x}) = f(\vec{x}_n - \vec{x}) + (\vec{x}' - \vec{x}_n) \cdot \vec{\nabla} f(\vec{x}_n - \vec{x})$$

$+$ subleading terms

$$\langle \eta \rangle = \sum_n \int d^3x' \, \eta_n(\vec{x}\,' - \vec{x}_n)\, f(\vec{x}_n - \vec{x}) +$$

$$+ \sum_n \int d^3x' \, \eta_n(\vec{x}\,' - \vec{x}_n)(\vec{x}\,' - \vec{x}_n) \cdot \vec{\nabla} f(\vec{x}_n - \vec{x}) =$$

$$= f(\vec{x}_n - \vec{x}) \sum_n \underbrace{\int d^3x' \, \eta_n(\vec{x}\,' - \vec{x}_n)}_{q_n} +$$

$$+ \sum_n \partial_i f(\vec{x}_n - \vec{x}) \underbrace{\int d^3x' \, \eta_n(\vec{x}\,' - \vec{x}_n)(x_i' - x_{i n})}_{d_{n,i}} =$$

Now we define:

$$\rho(\vec{x}) = \sum_n f(\vec{x}_n - \vec{x})\, q_n \rightarrow$$

$\rightarrow$ the average density

(it does not strongly depend on the detailed shape of f!) to check - integrate over $\vec{x}$.

$$\vec{P}(\vec{x}) = \sum_n \vec{d}_n \cdot f(\vec{x}_n - \vec{x})$$

$\downarrow$ electric polarization of the medium.

$$\langle \eta \rangle (\vec{x}) = \rho(\vec{x}) - \vec{\nabla}_x \cdot \vec{P}(\vec{x})$$

It is common to have $\rho = 0$, however $\vec{P}$ is generically non-zero, so we ignore the quadrupole terms.

Similar logic leads to

$$\langle \vec{j} \rangle = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M} \quad , \text{ where}$$

$\vec{J}(\vec{x})$ is the average current:

$$\vec{J}(\vec{x}) = \sum_n f(\vec{x} - \vec{x}_n) \, \vec{j}_n$$

Magnetization $\vec{M}(\vec{x})$ is defined as

$$\vec{M}(\vec{x}) = \sum_n \vec{m}_n \delta(\vec{x}_n - \vec{x})$$

$$\vec{m}_n = \frac{1}{2} \int d^3x' \, \vec{x}' \times \vec{j}_n(x')$$

Conservation equation, as other differential operators, commute with averaging:

$$0 = \frac{\partial}{\partial t} \langle \eta \rangle + \vec{\nabla} \cdot \langle \vec{j} \rangle = \frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{J}$$

At this stage Maxwell equations read:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{\vec{\nabla} \cdot \vec{P}}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{\partial \vec{P}}{\partial t} \mu_0 + \vec{\nabla} \times \vec{M} \mu_0 + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

If we define

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$ [Displacement]

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$ [H = magnetic field
B = induction (when in media)]

they simplify:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

The last two equations are simpler in terms of $\vec{E}$ and $\vec{B}$:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Before we treated charge distribution as given. Now we only give macroscopic quantities ρ and $\vec{J}$, because we can measure them. [Also called free charges]

Hence, in order to solve Maxwell equations, we need to determine $\vec{D}$ and $\vec{H}$ in terms of $\vec{E}$ and $\vec{B}$.

In many materials the relation is local and linear:

$$D_i = \varepsilon_{ij} E_j$$

$$H_i = [\mu_{ij}]^{-1} B_j$$

Moreover, if material is Isotropic

$$\varepsilon_{ij} \sim \delta_{ij} \quad \text{and} \quad \mu_{ij} \sim \delta_{ij} :$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

ε – electric permittivity of the material

μ – magnetic permeability of the material

Usually $\varepsilon > \varepsilon_0$, while for paramagnets $\mu > \mu_0$ and for diamagnets $0 < \mu < \mu_0$.
[see Landau–Lifshitz vol. 8]

ε, μ depend on the density of charges, temperature, spins, dipole moments of molecules, etc. They also in reality depend on $\vec{E}$ and $\vec{B}$, but we assume this dependence is weak and Taylor-expand it:

$$\varepsilon(E) = \varepsilon\Big|_{E=0} + \cancel{E\frac{\partial \varepsilon}{\partial E} + \dots}$$

small

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{vs} \qquad \vec{\nabla}\cdot\vec{E}_0 = \frac{\rho}{\varepsilon_0}$$

$$\vec{D} = \varepsilon\vec{E}$$

$|\vec{E}| < |\vec{E}_0|$ for the same ρ

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

Since $\frac{\mu}{\mu_0} - 1$ can have either sign, $\vec{B}$ can be weaker or stronger than in the vacuum (for the same $\vec{J}$).

Matching conditions

We would like to determine the conditions that occur on a boundary of two media with different μ and ε (one medium could be the vacuum). Let us use the integral form of the equations:

$$\vec{\nabla} \cdot \vec{D} = \rho \longrightarrow \int_{\partial V} d\vec{\sigma} \cdot \vec{D} = \int_V \rho(x) d^3x$$

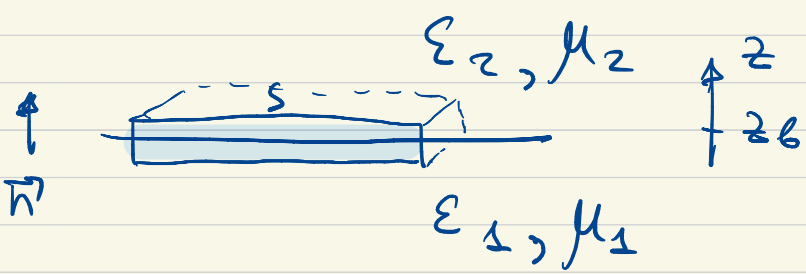
$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \longrightarrow \oint_{\partial S} d\vec{l} \cdot \vec{H} - \frac{\partial}{\partial t} \int_S d\vec{\sigma} \cdot \vec{D} =$$

$$= \int_S d\vec{\sigma} \cdot \vec{J}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \longrightarrow \oint_{\partial S} d\vec{l} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{\sigma} \cdot \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \int_{\partial V} d\vec{\sigma} \cdot \vec{B} = 0$$

Let's apply the first equation to a small volume near the boundary:



$$\rho = \rho_s \cdot \delta(z - z_0)$$

area

$$\int_{\partial V} d\vec{\sigma} \cdot \vec{D} = S\left(D_2^{\perp} - D_1^{\perp}\right) = \rho_s \cdot S \Rightarrow$$

$$\Rightarrow D_2^{\perp} - D_1^{\perp} = \rho_s$$

Same to eqn. 4:

$$B_2^{\perp} - B_1^{\perp} = 0$$

Now let's apply eqn 3 to an area crossing the boundary:

$$\oint_{\partial S} d\vec{l}\cdot\vec{E} = -\frac{d}{dt}\int_S d\vec{\sigma}\cdot\vec{B}$$

ε_2, μ_2

ε_1, μ_1

$$\int_{\partial S} d\vec{l}\cdot\vec{E} = L\left(\vec{E}_2^{\parallel} - \vec{E}_1^{\parallel}\right) = 0$$

$$\Downarrow$$

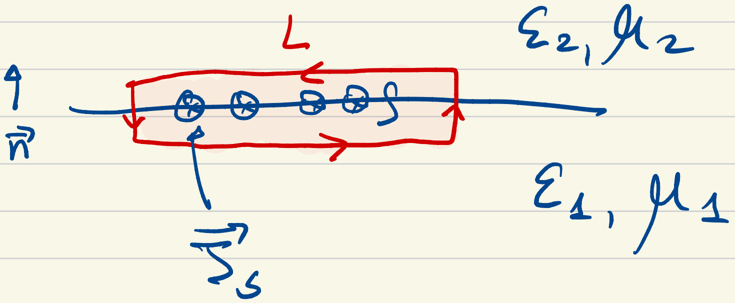
$\vec{E}^{\parallel}$ is continuous

$\int d\vec{\sigma}\cdot\vec{B}$ gives zero because there is no singular contribution

Finally, eqn. 2:

0 [No singular contribution]

$$\oint_{\partial S} d\vec{l}\cdot\vec{H} - \cancelto{0}{\frac{\partial}{\partial t}\int_S d\vec{\sigma}\cdot\vec{D}} = \int_S d\vec{\sigma}\cdot\vec{J}$$



(surface perpendicular to the surface current)

to the current

$$\oint_{\partial S} d\vec{l}\cdot\vec{H} = \left(H_2^{\parallel,\perp} - H_1^{\parallel,\perp}\right) L$$

$$\int d\vec{S}\cdot\vec{J} = L\cdot J_s$$

while for $H^{\parallel,\parallel}$ we get zero. In the vector notation:

$$\vec{H}_2^{\parallel} - \vec{H}_1^{\parallel} = \vec{J}_s \times \vec{n}$$